Extensions on the I-Cut-You-Freeze Protocol

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Abstract

In this paper we analyze several extensions to the I-cut-you-freeze protocol (Pegden et al., 2017), in which two parties take turns proposing districts, where the non-proposing party chooses one of those districts to keep, or "freeze." We attempt to introduce further constraints on the protocol that bring it closer to the current reality. Larger margins than 50.1% are needed for total confidence in one's ability to win a district, suggesting that constraints increasing the necessary confidence and introducing "coin flips" when the margins are not large enough for total confidence may bring the protocol closer to realistic implementation.

Keywords: Algorithmic Economics, Social Choice, Cake Cutting, Optimized Democracy, Gerrymandering, Redistricting

1. Introduction

In the United States of America, local representatives are elected to the House of Representatives via direct local elections held in single-member districts. Each district is roughly equal in population, and the winner is decided via plurality. The Congressional districts of a state are redrawn every ten years following the United States census. The rules for redistricting vary from state to state, but all draw new Congressional maps via either their own state legislature, redistricting commissions, or through some combination of the two former.

As one might imagine, there is a great deal of partisan interest in the drawing of these maps. A rather extreme (yet very politically relevant) example of this is that of the redrawing of Florida's congressional districts in 2022. In the state of Florida, the state legislature is to be responsible for redistricting proposals, which are then passed similarly to any other bill, to be approved by the governor prior to implementation. In the case of Florida, however, Republican Governor Ron DeSantis expressed general disapproval for proposals passed by the legislature, vetoing various proposals until the legislature expressed agreement regarding the consideration of a DeSantis-created proposal (one that shows a markedly favorable outcome in seats for the Republican Party)[1]. Such an event sets a dangerous precedent that may encourage other states to follow in Florida's footsteps.

Even in the case of "independent" redistricting commissions, there often exists some degree of partisan bias (even if this is justified through supposedly-equal representation in the commission from all relevant parties).

Due to the inherent partisan biases within the maps drawn for the legislature, there is no guarantee that the political makeup of the legislature will reflect the political composition of statewide vote casts in the election. For example, in the 2020 United States House of Representatives elections in Illinois, 57.10% of the population voted for the Democratic Party, yet Democratic representatives won 13 of the 18 district elections (72% of the seats). The Congressional map used in the election was drawn by the Illinois Democrats. Even worse, under the new map, Democrats are projected to win 14 of Illinois's 17 congressional seats after the 2022 election (82% of the seats). This phenomenon is known as *gerrymandering*.

Gerrymandering is the process of constructing districts in such a way as to give your political party an advantage in elections. There are two principal tactics in gerrymandering, "packing" and "cracking." Packing is when you create a single district that concentrates many voters of a single party in order to reduce that party's voting power in other districts. Cracking is when you spread out the voting power of a party across many districts in order to dilute their overall voting power in certain districts. The overall goal is to pack opposition voters into some districts, effectively guaranteeing them some small number of government seats, while cracking across other districts in order to raise the chances of one's own party gaining a high number of seats.

Gerrymandering in the field of Algorithmic Economics is generally viewed through the lens of cake cutting. Cake cutting tries to solve the problem of fairly dividing a heterogeneous, divisible good between several agents with different preferences. Naturally, one can see the connection to political districting and how cake cutting methods could be used to solve the problem of gerrymandering.

2. Related Works

Cake cutting aside, there are several metrics to detect gerrymandering in congressional district maps. Warrington (2019) compares and contrasts several metrics for detecting gerrymandering, such as the efficiency gap. The efficiency gap was first introduced by McGhee (2014) and Stephanopoulos and McGhee (2015), with the goal of the setting a quantifiable, legal standard for what constitutes gerrymandering. The metric works off the notion of wasted votes. A wasted vote is any vote cast for the losing party in a district, or the winning party above a majority. The efficiency gap is calculated as the difference between the two parties' wasted votes divided by the total number of votes. Most of the other metrics created are extensions on the efficiency gap, but ultimately, the efforts of this metric (and potentially others to come) failed when the Supreme Court rejected the standard in its 2018 decision in Gill v. Whitford.

In contrast to using methods to simply determine the fairness of districting proposals that were chosen under existing legal standards, there have been various protocols intro-

1. As of May 2022, the congressional map drawn by DeSantis' staff has been deemed unconstitutional by a Florida state judge.

duced that attempt to solve gerrymandering by finding a balance between two opposing partisan interests. The goal is to create a protocol by which there is no need to evaluate the fairness of a proposal, as the proposal is inherently fair to some degree prescribed by the protocol itself.

Landau et al. (2009) introduce a protocol that, following the nomenclature used by De Silva et al. (2018), we shall refer to as the LRY protocol. The overarching idea is to have one party divide one portion of a state into districts according to their own preferences, while the other party divides the rest of the state into districts with their own goals in mind. The thought is that having each party control some amount of the state yields a higher degree of fairness and satisfaction than other existing methods.

Issues arise, however, upon closer analysis of the LRY protocol. Through the analysis of De Silva et al. (2018) it is seen that the results achieved by this protocol can be arbitrarily far from the geometric target. Landau and Su (2014) introduce the geometric target as a measure of fairness and is commonly used as a relaxation to the proportionality metric.

Pegden et al. (2017) proposed a protocol with proven fairer outcomes. First, some notation. The state is defined as an interval $[0, n]$ (where n is the number of districts). An n-*districting* is a set of n disjoint subsets of the interval, each of size 1. The idea of votes in favor of one party is referred to as *loyalty*. If party A has 58% of the vote in a particular district, they are said to have a loyalty in that district of 0.58. The measure of the subset of $[0, n]$ loyal to Player 1 is denoted as s_1^n, and the measure of the subset of $[0, n]$ loyal to Player 2 is denoted as s_2^n. The *slate* of a player is the outcome from a districting, representing the number of districts loyal to them.

This protocol is structured such that two players take turns creating a n-districting and choosing which one of those n districts they want to keep, or "freeze." That district is then frozen, the roles swap, and the process continues, using the unfrozen area as the state.

The game resultant from this protocol can be represented recursively as follows, where $k = n$, $s_1 = s_1^n$, and $A = 1$, returning the slate of Player 1:

Procedure 1 I-cut-you-freeze Nongeometric

procedure $\text{GAME}_1(k, s_1, A)$ ▷ Player A divides first

  Player A chooses k numbers in $[0, 1]$: $x_{k,1}, \ldots, x_{k,k}$ s.t

$$\sum_{i=1}^{k} x_{k,i} = s_1$$

  Player B chooses an integer $i_k \in [k]$, where $\{A, B\} = \{1, 2\}$

  return $\text{GAME}_1(k - 1, s_1 - x_{k,i_k}, B) + [x_{k,i_k} \geq 0.5]$

end procedure

Let the unfrozen loyalties of Players 1 and 2 be $s_1^{(t)}$ and $s_2^{(t)}$, respectively, at the beginning of round t. The rounds begin at n and count down to 0, where the base case is defined as $\text{GAME}_1(0, 0, A) = 0$. **Note:** the procedure always returns the slate of Player 1, even if $A = 2$.

3. Our Contributions

Our contributions are inspired by the notion that mapmakers would want a more comfortable margin than 50.1% to consider a district safe. Generally, districts drawn in the United States of America are drawn such that they are solidly Republican or Democrat, fewer districts slightly lean one way or the other, and an even smaller portion are competitive. The interest in examining this model is to see how many solid districts are guaranteed under the I-cut-you-freeze protocol, given that players attempt to create them under optimal play.

3.1 Proposed Slate-Scoring Functions

We propose using one of the three following functions to score how likely a win is for Player 1 in a given district. The functions were derived based on the criticism of Pegden et al. (2017) that, in reality, partisan map drawers will attempt to increase the margin of victory for a given district well above the protocol's >0% margin of victory. Let a and b be the upper and lower bound for the ends of the margin of victory, these could be set to any with $[0, 1]$. Setting $a = b = 0.5$ will recover the original mechanism for the I-cut-you-freeze protocol. Function 1 is meant to be thought of as if the chance of winning a district is an even coin flip between a and b, where $b \geq a$.

$$h_1(x) = \begin{cases} 1 & \text{if } x \geq b \text{ ,} \\ 0.5 & \text{if } a < x < b \text{ ,} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

While function 2 is meant to be thought of as a weighted coin flip.

$$h_2(x) = \begin{cases} 1 & \text{if } x \geq b \text{ ,} \\ x & \text{if } a < x < b \text{ ,} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Function 3 is similar to function 2 in that it weights the chance of winning based on the loyalty of a district, but maps between zero and one versus a and b.

$$h_3(x) = \begin{cases} 1 & \text{if } x \geq b \text{ ,} \\ \frac{1}{b-a}(x-a) & \text{if } a < x < b \text{ ,} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Remark 1 *The above functions necessarily change the original meaning of the output of the I-cut-you-freeze procedure 1. The original meaning was the number of districts won by Player 1, whereas the new meaning is akin to a utility, or how optimally Player 1 is utilizing their loyalty. This is due in part to partial-seats being awarded to players when a competitive district is created. Thus, the above functions will produce a result that is less than or equal to what the original procedure would produce.*

3.2 A Higher Threshold for Guaranteed District Wins

We set $a = 0.425$ and $b = 0.575$. The assumption being made is that each player is only afforded a guaranteed win in a district if their loyalty within that district is at least 57.5%.

This emulates what are considered to be safe districts in American politics. Politicians generally draw up districts in which they have a lead of greater than or equal to 15 percentage points above the national average. So if the national average in America was a 2 point win by Republicans over Democrats, Democratic districts would need at least a 17 point margin of victory to be considered safe; that is, at least a 58.5% to 41.5% expected result. On the other hand, Republican districts would need a 13 point margin of victory to be considered safe. Assuming a relatively equal loyalty toward each party on the national level, the threshold can reasonably be set at 57.5%. Procedure 2 is a modification of procedure 1 using the aforementioned bounds and will be used from here on out unless otherwise specified.

Procedure 2 I-cut-you-freeze Nongeometric (Unweighted Competitive Districts)

procedure $\text{GAME}_2(k, s_1, A)$ ▷ Player A divides first

Player A chooses k numbers in $[0, 1]$: $x_{k,1}, \ldots, x_{k,k}$ s.t

$$\sum_{i=1}^{k} x_{k,i} = s_1$$

Player B chooses an integer $i_k \in [k]$, where $\{A, B\} = \{1, 2\}$

return $\text{GAME}_2(k-1, s_1 - x_{k,i_k}, B) + h_1(x_{k,i_k})$

end procedure

Let the function $f(k, s_1, A)$ be the output of $\text{GAME}_2(k, s_1, A)$ when the two players are playing optimally. Then we have that

$$f(k, s_1, 1) = \max_{x_{k,1},\ldots,x_{k,k}} \min_{i \in [k]} \left(f(k-1, s_1 - x_{k,i}, 2) + h_1(x_{k,i_k}) \right) \quad , \quad \text{and} \tag{4}$$

$$f(k, s_1, 2) = \min_{x_{k,1},\ldots,x_{k,k}} \max_{i \in [k]} \left(f(k-1, s_1 - x_{k,i}, 1) + h_1(x_{k,i_k}) \right) \quad . \tag{5}$$

This function is monotonically increasing with respect to s_1. This intuitively makes sense as with a higher total measure of loyalty it is expected that the slate attainable under optimal play will yield a greater than or equal slate than a lower total measure of loyalty. Given that the only change from Pegden et al. (2017) is increasing the threshold for which a district is to be considered confidently won, these results should not deviate. For convenience, lemma 3.1 from Pegden et al. (2017) is stated below. (The majority of the lemmas and overall proof structures shall resemble those of Pegden et al. (2017).)

Lemma 2 *$f(k, s_1, A) \leq f(k, s'_1, A)$ if $s_1 < s'_1$.* ■

The following lemma shows that under optimum play, players will divide the unfrozen region into districts with at most three distinct loyalty values.

Lemma 3 *For any $\text{GAME}_2(k, s_1, A)$, there are numbers $\omega \geq 0.575 > \tau > 0.425 \geq \lambda$ such that under optimum play, Player A will choose each $x_{k,i}$ to be ω, τ, or λ.*

Proof Let $W = \{i : x_{k,i} \geq 0.575\}$, $L = \{i : x_{k,i} \leq 0.425\}$, and $T = ([k] - W) - L$. First consider $A = 1$, applying lemma 2 to $\text{GAME}_2(k-1, s_1 - x_{k,i}, 2)$ one can see that the optimum move for Player 2 is to pick $i = \text{argmax}_{j \in W} x_{k,j}$, $i = \text{argmax}_{j \in T} x_{k,j}$, or $i = \text{argmax}_{j \in L} x_{k,j}$. This is because Player 2 will want to pack as much of Player 1's loyalty into a single district, regardless of outcome. In the instance of W, Player 2 is guaranteed to lose, therefore, they will attempt to waste as much of Player 1's loyalty in that district to prevent it from being used in others. (The reasoning is analogous for the case of T.) For the case of L, Player 2 is guaranteed a win, thus they will attempt to pack as many of Player 1's loyalty into the district while still remaining the winner, again this is about wasting the loyalty of Player 1 in this district so that it may not be utilized elsewhere. Therefore, under optimum player, Player 1 will assign an identical number ω to W, τ to T, and λ to L such that $\omega = \sum_{i \in W} x_{k,i}/|W|$ if $W \neq \emptyset$, $\tau = \sum_{i \in T} x_{k,i}/|T|$ if $T \neq \emptyset$, and $\lambda = \sum_{i \in L} x_{k,i}/|L|$ if $L \neq \emptyset$. This strategy eliminates the choice of Player 2 from wasting potential loyalties. The case for $A = 2$ is analogous. ■

In round t, if $s_1^{(t)} \geq 0.575t$, we say Player 1 is stronger and Player 2 is weaker in this round; if $s_1^{(t)} \leq 0.425t$, we say Player 2 is stronger and Player 1 is weaker in this round. If $s_1^{(t)} > 0.425t + 0.15$, we say Player 1 is α-competitive and Player 2 is β-competitive in this round; whereas, if $s_1^{(t)} \leq 0.425t + 0.15$, we say Player 1 is β-competitive and Player 2 is α-competitive in this round. (It is possible at $t = 1$ for both players to be β-competitive.) The full table of which label each player is at a given round is provided in table 1, and the scenarios that are possible are provided in table 2.

	Player A	
Label	Player 1	Player 2
Stronger	$s_1^{(t)} \geq 0.575t$	$s_1^{(t)} \leq 0.425t$
α-competitive	$s_1^{(t)} > 0.425t + 0.15$	$s_1^{(t)} < 0.575t - 0.15$
β-competitive	$s_1^{(t)} \leq 0.425t + 0.15$	$s_1^{(t)} \geq 0.575t - 0.15$
Weaker	$s_1^{(t)} \leq 0.425t$	$s_1^{(t)} \geq 0.575t$

Table 1: Labels for each player at the start of each round t.

	Player B			
Player A	Stronger	α-competitive	β-competitive	Weaker
Stronger	Impossible	Impossible	Impossible	Possible
α-competitive	Impossible	Impossible	Possible	Impossible
β-competitive	Impossible	Possible	Possible	Impossible
Weaker	Possible	Impossible	Impossible	Impossible

Table 2: Possibility of label pairs for the players.

From lemma 3, one can see that Player A's move is completely characterized by the choice of ω, τ, and λ. Therefore, Player B's response will result in the following three possibilities assuming $A = 1$ and without loss of generality: $f(k-1, s_1 - \omega, 2) + 1$, $f(k-1, s_1 - \tau, 2) + 0.5$, and $f(k-1, s_1 - \lambda, 2)$. This of course assumes the remaining game-play is optimal. This implies the following lemma.

Lemma 4 *Given possible choices* (ω, τ, λ) *and* $(\omega', \tau', \lambda')$ *for Player A satisfying* $\omega \leq \omega'$, $\tau \leq \tau'$, *and* $\lambda \leq \lambda'$, *the choice* (ω, τ, λ) *dominates the choice* $(\omega', \tau', \lambda')$ *if* $A = 1$*; otherwise, the choice* $(\omega', \tau', \lambda')$ *dominates the choice* (ω, τ, λ).

From the above, we can state the following two lemmas.

Lemma 5 *In any* $GAME_2(k, s_1, A)$*, Player A, if stronger or* β*-competitive, chooses* $x_{k,1} = x_{k,2} = \cdots = x_{k,k}$ *in optimum play. In particular:*

$$A \text{ stronger } \implies f(k, s_1, A) = f(k-1, s_1 - s_1/k, B) + h_1(s_1/k).$$

Proof This is the optimal move because in the strong case, Player A is only proposing districts for which they are equally solidly winning (preventing Player B from choosing the district that wastes Player A's loyalty). In the β-competitive case they force a competitive district (also preventing wasting Player A's loyalty) instead of handing the opponent a solid district. ■

Lemma 6 *In any* $GAME_2(k, s_1, A)$*, Player* $A = 1$*, if* α*-competitive, will choose the maximum amount of* $x_{k,i}$ *values to be 0.575 such that the remaining loyalty can be divided equally and still have a value greater than 0.425, let this value be* τ_1*. Player* $A = 2$*, if* α*-competitive, will choose the minimum amount of* $x_{k,i}$ *values to be 0.425 such that the remaining loyalty can be divided equally and still have a value less than 0.575, let this value be* τ_2*. In particular:*

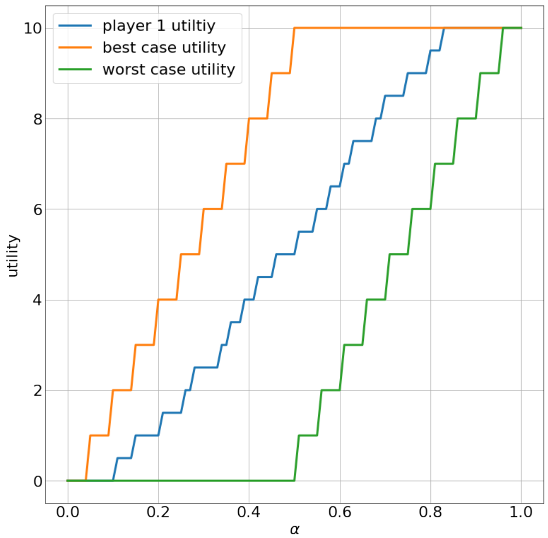
$$A = 1, \ \alpha\text{-competitive } \implies f(k, s_1, 1) = f(k-1, s_1 - \tau_1, 2) + 0.5.$$

$$A = 2, \ \alpha\text{-competitive } \implies f(k, s_1, 2) = f(k-1, s_1 - \tau_2, 1) + 0.5.$$

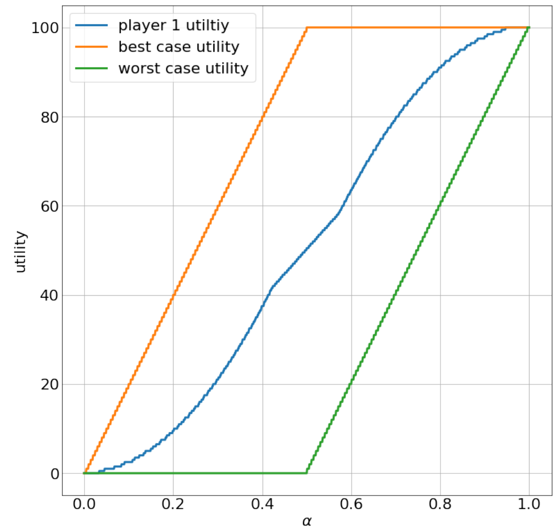
Proof This is the optimal move for both players as Player A is proposing districts that they either are equally solidly winning, or competitive without wasting loyalty (since it's score of 0.5 for both players, Player B would want to choose the competitive district that wastes the most of Player A's loyalty, but proposing competitive districts all of equal loyalty values eliminates the possibility of such a strategy for Player B). When A=2, player A wants to waste as much as possible of player B's vote, thus would want as many high loyalty value competitive districts as possible, which is achieved by minimizing the number of winning districts proposed (since player B will not choose the district with value 0.425, and will instead choose one of the competitive districts). ■

Lemma 7 *In any* $\mathrm{GAME}_1(k, s_1, A)$*, if Player A is weaker and* $s_A \geq 0.425$*, the following strategies for players are optimal:*

- *Let* $m = \left\lfloor \frac{1}{0.575} s_A \right\rfloor$ *if* $A = 1, m = \left\lceil \frac{1}{0.575} s_A \right\rceil - 1$ *if* $A = 2$*. Player A will divide the resources such that their proportion in each district is either 0 or* s_A/m*; however, this should be done only if* $s_A/m \geq 0.575$*, if not, use the following strategy.*



(a) 10 total seats.



(b) 100 total seats.

Figure 1: Simulated utility results for Player 1 under optimal play. α represents the loyalty of Player 1 in the state as a percentage.

- *Let* $m = \left\lceil \frac{1}{0.425} s_A \right\rceil - 1$ *if* $A = 1, m = \left\lfloor \frac{1}{0.425} s_A \right\rfloor$ *if* $A = 2$. *Player* A *will divide the resources such that their proportion in each district is either 0 or* s_A/m.

- *Player* B *will choose a district where his loyalty is 1.*

The previous lemma is what we believe to be optimal play when player A is weaker, though we were unable to prove this. Our intuition is that if a player is weaker they would want to pack their opponent into districts with 100 percent loyalty as many times as needed such that the remaining number of districts can each have loyalty equal to the total remaining loyalty during that round divided such that they are wins for themselves or competitive districts.

Figure 1 shows the resulting utility of Player 1 from the I-cut-you-freeze procedure using the modified procedure 2. Ideally we would like to prove a function for all cases. Interestingly the curve is not as smooth in the middle as the results of the original procedure (it becomes almost linear). It is important to recognize that the modified procedure also yields fair results in these empirical tests, meaning that our attempt to bring a bit more realism to the I-cut-you-freeze protocol is generally successful.

3.3 Weighted Competitive Districts

Next, we consider how the previous analysis might change if the competitive districts are weighted by the amount of loyalty within them. Procedure 3 is a modification of procedure 1 using the previous bounds and will be used from here on out unless otherwise specified. It is important to note that analysis using slate-scoring functions h_2 and h_3 is equivalent, as strategy is the same for each case.

Procedure 3 I-cut-you-freeze Nongeometric (Weighted Competitive Districts)

procedure GAME$_3(k, s_1, A)$ ▷ Player A divides first

Player A chooses k numbers in $[0, 1]$: $x_{k,1}, \ldots, x_{k,k}$ s.t

$$\sum_{i=1}^{k} x_{k,i} = s_1$$

Player B chooses an integer $i_k \in [k]$, where $\{A, B\} = \{1, 2\}$

return GAME$_2(k-1, s_1 - x_{k,i_k}, B) + h_2(x_{k,i_k})$

end procedure

The cases where Player A is stronger, weaker, and β-competitive remain the same. Strategy changes, however, in the case where Player A is α-competitive. Under the previous rules for competitive districts (each player gets 0.5 added to their slate), the overarching goal was to choose some amount of districts that they barely win, then all ties of the same value (see lemma 6). In any GAME$_3(k, s_1, A)$, Player $A = 1$, if α-competitive, will choose the minimum amount of $x_{k,i}$ values to be 0.575 such that the remaining loyalty can be divided equally and still have a value greater than 0.425, let this value be τ_1. Player $A = 2$, if α-competitive, will choose the maximum amount of $x_{k,i}$ values to be 0.425 such that the remaining loyalty can be divided equally and still have a value less than 0.575, let this value be τ_2. Under the previous rules, Player A tried to minimize the loyalty value (in terms of their loyalty) of the ties to waste Player B's loyalty. Under these new rules, Player A tires to maximize the loyalty value of the competitive districts to give themselves the highest value competitive districts possible (since the value added to each player's slate is equal to their loyalty in the frozen competitive district, as opposed to giving each player 0.5 regardless).

4. Future Work

Some other possible extensions to Pegden et al. (2017) are imposing geometric constraints on the geometric procedure. Pegden et al. (2017), propose the constraint $4\pi A_D/P_D$, where A_D is the area of a district and P_D is the perimeter. Another such constraint is imposing that a district must be a fat object, which has no greater a "slimness factor" than R. Simply put, the district must be R-fat. Geometric constraints already exist for redistricting, for instance, the district must be a contiguous object and generally there is this notion that the district must be "compact;" however, there are no lawful definitions of "compact." Thus, imposing such compactness constraints may not make much sense, as there is little use of such a constraint in common redistricting practice. This was also mentioned in brief by Pegden et al. (2017). Regardless, some interesting results may arise from imposing such constraints; for example, it may be possible for a player to dominate the remainder of the game by abusing the geometric constraints imposed in the initial phase of the game. Such results would provide greater insight to the I-cut-you-freeze protocol.

5. Conclusion

The I-cut-you-freeze protocol markedly improves upon the previous work done to prevent gerrymandering, as it provably yields results with a high degree of fairness. Notably, it also removes the need for an impartial party to run the procedure, as was the case in the LRY protocol. Additionally, it directly attempts to stop gerrymandering as opposed to simply using statistical metrics to determine if a map has been gerrymandered, and to what extent. Pegden et al. (2017) proved the fairness of the protocol with relatively naive constraints on the strategies of the players. This paper shows that even with more realistic constraints on the procedure, the I-cut-you-freeze protocol remains a partisan districting protocol with provably nonpartisan outcomes.

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Appendix

The python code used to generate the two plots.

```
import matplotlib.pyplot as plt
import numpy as np

plt.style.use("_mpl-gallery")
plt.rcParams.update({"font.size": 22})


def is_stronger(player, remaining_loyalty, t):
    if player == 1:
        return remaining_loyalty >= 0.575 * t
    else:
        return remaining_loyalty <= 0.425 * t


def is_alpha_competitive(player, remaining_loyalty, t):
    if player == 1:
        return remaining_loyalty > 0.425 * t + 0.15
    else:
        return remaining_loyalty < 0.575 * t - 0.15


def is_beta_competitive(player, remaining_loyalty, t):
    if player == 1:
        return remaining_loyalty <= 0.425 * t + 0.15
    else:
        return remaining_loyalty >= 0.575 * t - 0.15


def is_weaker(player, remaining_loyalty, t):
    if player == 1:
        return remaining_loyalty <= 0.425 * t
    else:
        return remaining_loyalty >= 0.575 * t


def h(x):
    if x >= 0.575:
        return 1
    elif x > 0.425:
        return 0.5
    return 0


def get_x_alpha_competitive_player_1(remaining_loyalty, k):
    z = k - 1
    while z > 1 and (remaining_loyalty - 0.575 * z) / (k - z) <= 0.425:
        z -= 1
    return (remaining_loyalty - 0.575 * z) / (k - z)


def get_x_alpha_competitive_player_2(remaining_loyalty, k):
    z = k - 1
```

```
    while z > 1 and (remaining_loyalty - 0.425 * z) / (k - z) >= 0.575:
        z -= 1
    return (remaining_loyalty - 0.425 * z) / (k - z)


def procedure(k, remaining_loyalty, player_a):

    if k == 0:
        return 0

    if is_stronger(player_a, remaining_loyalty, k):
        x = remaining_loyalty / k
    elif is_alpha_competitive(player_a, remaining_loyalty, k):
        if player_a == 1:
            x = get_x_alpha_competitive_player_1(remaining_loyalty, k)
        else:
            x = get_x_alpha_competitive_player_2(remaining_loyalty, k)
    elif is_weaker(player_a, remaining_loyalty, k):
        x = 0 if player_a == 1 else 1
    elif is_beta_competitive(player_a, remaining_loyalty, k):
        x = remaining_loyalty / k

    return procedure(
        k - 1, remaining_loyalty - x, 2 if player_a == 1 else 1
    ) + h(x)


seats = 100

xs = np.arange(seats * 10 + 1) / 10
ys = []

for i in xs:
    ys.append(procedure(seats, i, 1))

# plot
fig, ax = plt.subplots(figsize=(10, 10))

ax.plot(xs, ys, linewidth=3.0, label="player 1 utiltiy")

best = np.floor(2 * xs)
mask = best > seats
best[mask] = seats

ax.plot(xs, best, linewidth=3.0, label="best case utility")
ax.plot(xs, np.flip(seats - best), linewidth=3.0, label="worst case utility"
    )

ax.set(xlabel=r"$\alpha$", ylabel="utility")

plt.legend()
plt.show()
```